Hedging Derivative Securities with VIX Derivatives: A Discrete-Time $\varepsilon$-Arbitrage Approach

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Abstract

This paper is an extension of the methodology proposed by Bertsimas et al. [1] to solve an optimal-replication problem for derivative securities trading in an incomplete market environment. We include an additional volatility derivative (e.g. a VIX futures contract) in the replicating portfolio and thus a new discrete-time self-financing dynamic portfolio strategy, which uses the underlying securities, bonds and volatility derivatives is proposed to approximate the payoff of a derivative security at maturity. Recursive expressions are derived for the stochastic dynamic programming applied to minimize the mean-squared-error loss function, which can be readily applied in practice with pre-calculated parameter matrices. Using the replication error ($\varepsilon$) and the relative replication error relative to the replication cost, we investigate the effectiveness of using volatility derivatives as the additional hedging instrument, under a stochastic volatility model as the return-generating process. Results are compared with the original optimal replication strategy proposed by Bertsimas et al. [1] and the delta hedging strategy proposed by Black and Scholes [2]. Both the replication error and the relative replication error of the new strategy are found being smaller and less sensitive to the volatility of the volatility.

Keywords

Optimal replication, dynamic portfolio, incomplete market, discrete time, stochastic volatility model
1. INTRODUCTION

The introduction of the volatility index or the sigma index by Brenner and Galai [3] created the idea of using volatility product to directly hedge the volatility exposure of derivatives in the market. This was made feasible when the Chicago Board of Options Exchange (CBOE) launched the VIX index in 1993, which was later revised in 2003 (see [4]).

The idea of hedging derivatives began when Black and Scholes [2] and Merton [5] developed the option-pricing formula using the delta hedging strategy and the no-arbitrage argument. If we are able to replicate the payoff function of a derivative security using other financial securities exactly, the initial cost of the financial securities used in the replication should be the price of the derivative security. However, the delta hedging approach is no longer optimal when the assumption on continuous trading is violated, which is the case in real life. To tackle this problem, Bertsimas et al. [1], Schweizer [6] and Schweizer [7] developed optimal-replication strategies using dynamic programming. The replicating portfolio used in Bertsimas et al. [1] includes stocks and bonds, and only needs one control parameter. The computation of the optimal control parameter for the number of shares of stocks plays a similar role as the adjustment in the number of shares of stocks in the delta hedging strategy. A natural extension to their work is to include gamma hedging, vega hedging or rho hedging into the optimal-replication strategy. However in the literature, there is a lack of research work in this direction.

In this paper, we hope to fill in this gap and explore additional properties that the vega hedging can bring into the replicating portfolio. We attempt to extend the optimal-replication strategy by including a volatility derivative into the replicating portfolio. Volatility can be readily and liquidly traded in exchanges, for example the VIX for the S&P500 index, VXN for the NASDAQ 100 index and the VXD for the Dow Jones Industrial Average index. Similar to the work done by Merton [5], we seek to create a self-financing portfolio at time 0 that will replicate the pay-off of a derivative security at maturity using the dynamic hedging strategy. Under a discrete and finite trading framework, exact replication is almost impossible. This is unlike the situation in Black and Scholes [2], where trading is assumed to be continuous. Therefore, the optimisation problem is defined based on the mean-squared error loss function ($\varepsilon$ is used to denote the root-mean-squared-error). It is noted that using a different error function will lead to a different optimal-replication strategy. In Bertsimas et al. [1], it has been elaborated “Why Mean-Squared Error” is used. If every financial institution believes in minimizing the mean-squared loss function, the initial portfolio value $V_0^*$ can be viewed as the price of the derivative security.

To demonstrate the practical use of the new vega-hedging optimal replication strategy, an example based on a simulated price path is given in Table 1. The
strategy that has a zero portfolio value at maturity is considered to be the best approach. It is noticed that, the new optimal-replication strategy ($V_T = -6.7698$) outperforms the other two strategies namely [2] delta hedging strategy ($V_T = -7.0053$) and [1] optimal-replication strategy ($V_T = -7.2829$). The example is only meant to be illustrative but not conclusive as this showcases only one price path. Note that the optimal replication strategy is defined in the mean-square sense and uses expectations. A more extensive comparison study, using Monte-Carlo simulations and 10,000 sample paths, will be provided.

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Table 1: Comparison of optimal-replication strategies with and without vega hedging and the Black-Scholes delta hedging strategy for a 6-month at-the-money put option on 1000 shares of a $1-stock over 25 trading intervals.

In section 2, we define the new optimal replication problem and propose a solution to the problem. Section 3 solves the optimal replication problem using numerical methods and investigates the effectiveness of the new replication strategy by comparing it with the other two strategies ([1] and [2]). Conclusion is made in section 4.

2. \( \varepsilon \)-ARBITRAGE STRATEGIES

In this section, we extend the work done by Bertsimas et al. [1] by including a volatility derivative into the self-financing portfolio. In section 2.1, we formulate the
new optimal-replication problem and derive the self-financing equation. In section 2.2 we propose a solution for the optimal-replication problem in discrete time via stochastic dynamic programming.

2.1. The Optimal-Replication Problem

Consider a derivative security with a pay-off function $F(P_T; Z_T)$ at maturity $T$, where $P_t$ denotes the price of the underlying asset for $0 \leq t \leq T$ and $Z_t$ denotes other state variables. We want to use a self-financing portfolio and a dynamic portfolio strategy consisting of stocks, bonds and volatility derivatives such that, at time $T$, we will be able to replicate the pay-off $F(P_T; Z_T)$ as closely as possible in a mean-squared sense. For a more precise formulation of the optimal-replication problem, we have the following assumptions:

Assumption 1: Markets are frictionless, i.e., there are no taxes, transactions costs, short-sales restrictions and borrowing restriction.

Assumption 2: The risk-less borrowing and lending rate is $r$.

Assumption 3: There exists a finite-dimensional vector $Z_t$ of state variables whose components are not perfectly correlated with the prices of any traded securities, and $[P_t, Z_t]$ is a vector Markov process.

Assumption 4: Trading takes place at known fixed times $t \in \mathcal{S}$. If $\mathcal{S} = \{t_0, t_1, \cdots, t_N\}$, trading is said to be discrete. If $\mathcal{S} = [0, T]$, trading is said to be continuous.

Assumption 5: Volatility can be liquidly traded in an exchange in the form of futures or options and the traded volatility is a 1-day forward looking implied volatility.

Note that the first four assumptions are the same as those given in [1]. The additional assumption, i.e., Assumption 5, is a strong assumption on the traded volatility. In practice, volatility indexes in exchanges such as VIX and VHSI are 30-day forward looking implied volatilities.

At time $t = t_0 = 0$, consider creating a portfolio of stocks (number of shares denoted by $\theta_{1,t_0}$), bonds (in dollars denoted by $B_{t_0}$) and entering a short position in volatility futures (number of contracts denoted by $\theta_{2,t_0}$ and price quoted as $\sigma_{t_0}$), all together at a cost $V_0(= V_{t_0})$. As time progresses, the market value of the portfolio at time $t \in [t_0, t_1)$ will be
\[ V_t = \theta_{1,t_0} P_t + B_t + \theta_{2,t_0} (\sigma_t - \sigma_{t_0}). \]  

Note that, at \( t = t_0, \)

\[ V_{t_0} = \theta_{1,t_0} P_{t_0} + B_{t_0}. \]

For any time between \( t_0 \) and \( t_1 \), the daily settlement of the futures contract can be taken care of by changing the value of \( B_t \). When it is time \( t_1 \), we determine a new set of parameter values (\( \theta_{1,t_1} \) and \( \theta_{2,t_1} \)). We exit in full the previous position in futures and enter a new one. Being a self-financing portfolio strategy, we have

\[ P_{t_1} (\theta_{1,t_1} - \theta_{1,t_0}) + (B_{t_1} - B_{t_0}) + \theta_{2,t_0} (\sigma_{t_1} - \sigma_{t_0}) = 0. \]

This implies that, at discrete times \( \{t_0, t_1, \cdots, t_N\} \), we have

\[ V_{t_{i+1}} - V_{t_i} = \theta_{1,t_i} (P_{t_{i+1}} - P_{t_i}) + \theta_{2,t_i} (\sigma_{t_{i+1}} - \sigma_{t_i}), \quad i = 0, 1, \cdots, N - 1. \]

We seek a self-financing portfolio strategy \( \{\theta_{1,t_i}, \theta_{2,t_i}\}, \quad i = 0, 1, \cdots, N - 1 \), such that the terminal value \( V_T \) is as close as possible to the payoff of the derivative at maturity.

### 2.2. \( \varepsilon \)-Arbitrage in Discrete Time

In this section, we solve the discrete optimal-replication problem using stochastic dynamic programming. Only the one-dimensional state variable \( Z_t = Z_{t_i} = \sigma_{t} = \sigma_{t_i} \) is considered. The usual cost-to-go or value function \( J_i \) (at \( t = t_i \)) is defined as

\[ J_i(V_i, P_i, \sigma_i) = \min_{\theta_{1,i}, \theta_{2,i}} E \{ |V_N - F(P_N, \sigma_N)|^2 \}, \]

where \( P_i = P_{t_i}, \) \( V_i = V_{t_i}, \) \( \sigma_i = \sigma_{t_i} \) are state variables, \( \theta_{1,i} = \theta_{1,t_i} \) and \( \theta_{2,i} = \theta_{2,t_i} \) are control variables. By applying Bellman’s principle of optimality recursively (see for example Bertsekas et a. [8]),

\[ J_N(V_N, P_N, \sigma_N) = |V_N - F(P_N, \sigma_N)|^2, \]

\[ J_i(V_i, P_i, \sigma_i) = \min_{\theta_{1,i}, \theta_{2,i}} E \{ J_{i+1}(V_{i+1}, P_{i+1}, \sigma_{i+1}) | V_i, P_i, \sigma_i \}, \]
Schweizer [6] has provided sufficient conditions for the existence of the optimal hedging strategy. In particular, under Assumptions 1-5 and the self-financing condition given in Equation (4), the solution of the optimal-replication problem can be characterized by the following:

1) The value function $J_t(V_t, P_t, \sigma_t)$ is quadratic in $V_t$, i.e., there exist functions $a_t(P_t, \sigma_t), b_t(P_t, \sigma_t)$ and $c_t(P_t, \sigma_t)$ such that

$$ J_t(V_t, P_t, \sigma_t) = a_t(P_t, \sigma_t)[V_t - b_t(P_t, \sigma_t)]^2 + c_t(P_t, \sigma_t). \quad (8) $$

2) Both the optimal controls $\theta^*_t(V_t, P_t, \sigma_t)$ and $\theta^*_t(V_t, P_t, \sigma_t)$ are linear in $V_t$, i.e.,

$$ \theta^*_t(V_t, P_t, \sigma_t) = p_{1,t}(P_t, \sigma_t) - V_t \cdot q_{1,t}(P_t, \sigma_t), \quad (9) $$

$$ \theta^*_t(V_t, P_t, \sigma_t) = p_{2,t}(P_t, \sigma_t) - V_t \cdot q_{2,t}(P_t, \sigma_t). \quad (10) $$

3) The functions $a_t(\cdot), b_t(\cdot), c_t(\cdot), p_{1,t}(\cdot), p_{2,t}(\cdot), q_{1,t}(\cdot)$ and $q_{2,t}(\cdot)$ are defined recursively as

$$ a_N(P_N, \sigma_N) = 1, \quad (11) $$

$$ b_N(P_N, \sigma_N) = F(P_N, \sigma_N), \quad (12) $$

$$ c_N(P_N, \sigma_N) = 0, \quad (13) $$

and for $i = N - 1, \cdots, 1, 0$, we have the following formula. (Note that, to simplify the notations, we denote $E_t[\cdot] = E[\cdot | (P_t, \sigma_t)], a_i = a_i(P_t, \sigma_i)$ and so on.)

$$ p_{1,t} = \frac{E_t[a_{i+1} \cdot (\Delta \sigma_t)^2] \cdot E_t[a_{i+1} \cdot b_{i+1} \cdot \Delta P_t] - E_t[a_{i+1} \cdot \Delta P_t \cdot \Delta \sigma_t] \cdot E_t[a_{i+1} \cdot b_{i+1} \cdot \Delta \sigma_t]}{E_t[a_{i+1} \cdot (\Delta P_t)^2] \cdot E_t[a_{i+1} \cdot (\Delta \sigma_t)^2] - (E_t[a_{i+1} \cdot \Delta P_t \cdot \Delta \sigma_t])^2}, \quad (14) $$

$$ q_{1,t} = \frac{E_t[a_{i+1} \cdot (\Delta \sigma_t)^2] \cdot E_t[a_{i+1} \cdot \Delta P_t] - E_t[a_{i+1} \cdot \Delta P_t \cdot \Delta \sigma_t] \cdot E_t[a_{i+1} \cdot \Delta \sigma_t]}{E_t[a_{i+1} \cdot (\Delta P_t)^2] \cdot E_t[a_{i+1} \cdot (\Delta \sigma_t)^2] - (E_t[a_{i+1} \cdot \Delta P_t \cdot \Delta \sigma_t])^2}, \quad (15) $$

$$ p_{2,t} = \frac{E_t[a_{i+1} \cdot (\Delta P_t)^2] \cdot E_t[a_{i+1} \cdot b_{i+1} \cdot \Delta \sigma_t] - E_t[a_{i+1} \cdot \Delta P_t \cdot \Delta \sigma_t] \cdot E_t[a_{i+1} \cdot b_{i+1} \cdot \Delta \sigma_t]}{E_t[a_{i+1} \cdot (\Delta P_t)^2] \cdot E_t[a_{i+1} \cdot (\Delta \sigma_t)^2] - (E_t[a_{i+1} \cdot \Delta P_t \cdot \Delta \sigma_t])^2}, \quad (16) $$

$$ q_{2,t} = \frac{E_t[a_{i+1} \cdot (\Delta P_t)^2] \cdot E_t[a_{i+1} \cdot \Delta P_t] - E_t[a_{i+1} \cdot \Delta P_t \cdot \Delta \sigma_t] \cdot E_t[a_{i+1} \cdot \Delta \sigma_t]}{E_t[a_{i+1} \cdot (\Delta P_t)^2] \cdot E_t[a_{i+1} \cdot (\Delta \sigma_t)^2] - (E_t[a_{i+1} \cdot \Delta P_t \cdot \Delta \sigma_t])^2}, \quad (17) $$

for $i = 0, 1, \cdots, N - 1$. 
\[ a_i = E_i \left[ a_{i+1} \cdot (1 - q_{1,i} \cdot \Delta P_t - q_{2,i} \cdot \Delta \sigma)^2 \right], \quad (18) \]

\[ b_i = \frac{1}{a_i} \cdot E_i \left[ a_{i+1} \cdot (1 - q_{1,i} \cdot \Delta P_t - q_{2,i} \cdot \Delta \sigma) \cdot (b_{i+1} - p_{1,i} \cdot \Delta P_t - p_{2,i} \cdot \Delta \sigma) \right], \quad (19) \]

\[ c_i = E_i [c_{i+1}] + E_i \left[ a_{i+1} \cdot (b_{i+1} - p_{1,i} \cdot \Delta P_t - p_{2,i} \cdot \Delta \sigma)^2 \right] + a_i \cdot (b_i)^2, \quad (20) \]

where \( \Delta P_t = P_{t+1} - P_t \) and \( \Delta \sigma = \sigma_{i+1} - \sigma_i \).

4) Given the optimal controls, \( \theta_1^* \) and \( \theta_2^* \), the minimum replication error as a function of the initial wealth \( V_0 \) is

\[ J_0(V_0, P_0, \sigma_0) = a_0 \cdot (V_0 - b_0)^2 + c_0. \quad (21) \]

Hence the initial wealth that minimizes the replication error is \( V_0^* = b_0(P_0, \sigma_0) \) and its corresponding replication error \( \epsilon^* = \sqrt{c_0(P_0, \sigma_0)} \).

3. NUMERICAL ANALYSIS

We analyse the effectiveness of vega hedging using the optimal-replication strategy by observing its numerical solutions in graphs and tables. We use the same stochastic volatility model given in [1]. That is,

\[
\begin{align*}
dP_t &= \mu \cdot P_t \cdot dt + \sigma_t \cdot P_t \cdot dW_t^p \\
d\sigma_t &= -\delta \cdot \sigma_t \cdot (\sigma_t - \zeta) \cdot dt + \kappa \cdot \sigma_t \cdot dW_t^\sigma
\end{align*}
\]

The numerical solution procedure is similar to what was given in [1]. In section 3.1, we describe the numerical procedure. And to study the sensitivity of the solution to parameters, as what was done in Bertsimas et al. [1], we present the plots, side by side, for the replication cost, the replication error and the relative replication error, respectively from the original optimal-replication method and the new one, with some of the parameters varying. In section 3.2, we use a comparison study to showcase the effectiveness of the new optimal-replication strategy with vega hedging.

3.1. The Numerical Procedure
To implement the recursive solution procedure formulated in Equations (14)-(20), we first represent the functions $a(P, \sigma), b(P, \sigma), c(P, \sigma), p_1(P, \sigma), p_2(P, \sigma), q_1(P, \sigma)$ and $q_2(P, \sigma)$ with their respective spatial grid $\{(P^j, \sigma^k): j = 0, 1, \ldots, J; k = 0, 1, \ldots, K\}$. For any given $(P, \sigma)$, we use a piece-wise linear interpolation to provide the approximation. The values $a_i(P^j, \sigma^k), b_i(P^j, \sigma^k), c_i(P^j, \sigma^k), p_{1,i}(P^j, \sigma^k), p_{2,i}(P^j, \sigma^k), q_{1,i}(P^j, \sigma^k),$ and $q_{2,i}(P^j, \sigma^k)$ are updated recursively backwards through time according to Equations (14)-(20).

Same as what was used in Bertsimas et al. [1], the expectations in Equations (14)-(20) are evaluated by replacing them with the corresponding integrals. When evaluating these integrals, the Gauss-Hermite quadrature formula (see Stroud[9]) is used. All the examples used in this paper are for an at-the-money European put option with a six-month maturity. 25 equal trading intervals are used, i.e. $t_0 = 0$ and $\Delta t = t_{i+1} - t_i = \frac{1}{50}$ for $i = 0, 1, \ldots 49$. (Note that this is same setting used by Bertsimas et al. [1]).

The dynamics of the stock price and volatility are described by:

\begin{equation}
\begin{align*}
P_{i+1} &= P_i \cdot \exp \left( \left( \mu - \frac{\sigma^2_i}{2} \right) \Delta t + \sigma_i \cdot \sqrt{\Delta t} \cdot z^{P}_i \right) \\
\sigma_{i+1} &= \sigma_i \cdot \exp \left( -\delta \cdot (\sigma_i - \zeta) - \frac{\kappa^2}{2} \right) \cdot \Delta t + \kappa \cdot \sqrt{\Delta t} \cdot z^{\sigma}_i
\end{align*}
\end{equation}

where $z^{P}_i, z^{\sigma}_i \sim \mathcal{N}(0,1)$ and $E[z^{P}_i z^{\sigma}_i] = \rho$. The parameters of the model are chosen to be:

\begin{equation}
\mu = 0.07, \sigma_0 = 0.13, \zeta = 0.153, \delta = 2, \kappa = 0.4, \rho = 0.6.
\end{equation}

To study the sensitivity of the solution to parameter values and to provide a comparison to the original optimal-replication strategy, we plot in Figures 1 to 3 to show, respectively, the replication cost, the replication error and the relative replication error. We have the following observations.

- The replication costs computed by the two optimal-replication strategies are similar. In both strategies, the replication cost is most sensitive to the initial volatility ($\sigma_0$).
- In the new optimal-replication strategy, the replication error and the relative replication error are smaller than the original optimal-replication strategy.
- The new optimal-replication strategy becomes less sensitive to the volatility of volatility ($\kappa$), but more sensitive to the reverted mean value ($\zeta$).
Figure 1: The difference between the replication cost and the intrinsic value of a 6-month maturity European put option, plotted as a function of the initial stock price. In these panels, $\zeta$, $\delta$, $\kappa$ and $\sigma_0$ are each varied, respectively, while the other parameter values are held fixed. The variation in each parameter is obtained by multiplying its original value by 1.5 (dashed-dotted line) and 0.5 (dashed line). The original optimal-replication strategy is denoted by SV-D, and the new optimal-replication strategy proposed in this paper is denoted by SV-DV.
Figure 2: The replication error of a 6-month maturity European put option, plotted as a function of the initial stock price. In these panels, $\zeta$, $\delta$, $\kappa$ and $\sigma_0$ are each varied, respectively, while the other parameter values are held fixed. The variation in each parameter is obtained by multiplying its original value by 1.5 (dashed-dotted line) and 0.5 (dashed line). The original optimal-replication strategy is denoted by SV-D, and the new optimal-replication strategy proposed in this paper is denoted by SV-DV.
Figure 3: The relative replication error of a 6-month maturity European put option (relative to the replication cost), plotted as a function of the initial stock price. In these panels, $\zeta$, $\delta$, $\kappa$, and $\sigma_0$ are each varied, respectively, while the other parameter values are held fixed. The variation in each parameter is obtained by multiplying its original value by 1.5 (dashed-dotted) and 0.5 (dashed line). The original optimal-replication strategy is denoted by SV-D, and the new optimal-replication strategy proposed in this paper is denoted by SV-DV.
3.2. Numerical Comparison

In this section, we perform a comparison study among the three dynamic portfolio strategies mentioned in this paper, i.e., the delta-hedging strategy by Black and Scholes [2], the original optimal-replication strategy by Bertsimas et al. [1] and the new optimal-replication strategy with vega hedging. 10,000 different price paths (and volatility paths) are simulated. The comparison result is given in Table 2. We note that, there are 5526 cases (55.26%) that the new strategy produces the smallest replication error. In the comparison between each two strategies, the original optimal-replication strategy beats the Black-Scholes delta-hedging strategy by 5652 to 4348. The new optimal-replication strategy beats the other two strategies, respectively, by 6842 to 3158 and by 6468 to 3532.

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<td>4348</td>
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Table 2: Comparison of the three strategies over 10,000 simulations.

4. CONCLUSION

We have successfully extended the original optimal-replication strategy proposed by Bertsimas et al. [1] by including an additional volatility derivative, from one control parameter to two control parameters. Recursive solution procedure is derived and solved under the stochastic volatility model (used as the return-generating process). By comparisons, the new strategy with vega hedging is found to be a better approach, compared to the Black and Scholes delta-hedging strategy and the Bertsimas et al. optimal-replication strategy. The new strategy is found to be less sensitive to the volatility of volatility ($\kappa$).

References


